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Synchronization of chaotic orbits: The effect of a finite time step

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Two chaotic orbits can be synchronized by driving one of them by the other. Some of the variables of the driven orbit are set continuously to the corresponding variables of the drive orbit. It has been seen that synchronization can be achieved if the subsystem Lyapunov exponents corresponding to the remaining or response variables are all negative. We find that a procedure where the drive variable is set at discrete times can also achieve synchronization. However, the synchronization criterion is altered by the effect of the drive being set at finite time steps. An important consequence of this is found in the Lorenz system where synchronization can be achieved with z as the drive variable despite the existence of a marginal subsystem Lyapunov exponent. We also find that synchronization can be achieved for the Rössler attractor with z as the drive, even though the largest subsystem Lyapunov exponent is positive. In addition, we find that there is an optimal time step corresponding to the fastest rate of convergence for both cases above. Our synchronization criterion reduces to the usual subsystem-Lyapunov-exponent criterion in the limit of the time step tending to zero.

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I. INTRODUCTION

The problem of the control of nonlinear dynamical systems is a topic of much current interest [1–6]. This problem is particularly interesting when the desired trajectory is in the chaotic regime. In the case of chaotic systems, a freely evolving trajectory cannot be reproduced due to the sensitive dependence on initial conditions and our inability to set the initial conditions precisely. Pecora and Carroll [1,4] have devised an ingenious method for forcing a desired chaotic trajectory onto a system by the use of appropriate drive variables. Some of the variables of the desired trajectory chosen to be the drive variables and the corresponding variables of the evolving system are continuously set to match this drive. The remaining variables, called the response variables, are allowed to evolve freely, under the equations of motion of the system. Pecora and Carroll [1] have shown that if the drive variables are such that the subsystem Lyapunov exponents (SLE's) corresponding to the remaining or response variables are all negative, then the response variables are controlled by the drive variables and all variables of the system settle down onto the desired chaotic trajectory.

As mentioned above, the Pecora-Carroll method involves the setting of the drive variables in a continuous fashion. However, in some cases, it may be impossible to set the drive variable continuously. In others, the setting of the drive variable at discrete time intervals may prove to be more cost effective than a continuous setting. It is thus useful to have a variant of the Pecora and Carroll method wherein the drive variable is set at discrete time intervals. We propose such a variant. A significant point of difference between this method and that of Pecora and Carroll is that when the drive is set at discrete time steps, the drive variables evolve freely between two settings. Thus even the drive variables tend to drift away from the desired orbit in the finite time interval during two settings. This difference has an important consequence. Synchronization can be achieved for some cases where the subsystem Lyapunov exponents of the response system do not satisfy the criterion of negativity. This is due to the fact that the criterion for synchronization is itself modified by the finite time procedure.

We study the synchronization procedure using the method of finite time step in this paper. We obtain the solution to the driven evolution with the finite time step within the local linear approximation and use this to obtain the finite-time-step criterion for synchronization. We show that our criterion reduces to that of Pecora and Carroll in the limit of continuous evolution. The advantages of the finite-time-step method are seen in the case of the Lorenz and Rössler systems. We find that synchronization can be obtained with z as the drive variable despite the fact that the largest subsystem Lyapunov exponent is marginal. A similar result is found for the Rössler attractor with a z drive where synchronization can be achieved in spite of the presence of a positive subsystem Lyapunov exponent. In addition, we find that there is an optimal value of the time step τ for which the length of the transient is the minimum.

II. EVOLUTION FOR FINITE TIME STEP

Consider an autonomous n-dimensional dynamical system evolving via the evolution equation

$$\dot{u} = f(u,\mu) , \tag{1}$$

where

$$u = (u_1, u_2, \ldots, u_n) ,$$

$$f(u,\mu) = (f_1(u,\mu), \ldots, f_n(u,\mu))$$

are n-dimensional vectors and the function f depends on the set of parameters μ. The parameters μ are such that the trajectories of the system lie on a chaotic attractor.

We wish to force the system onto a desired chaotic orbit. We start the procedure of synchronization [1,4] by dividing the variables of the system into two subsystems, a drive subsystem u_d and a response subsystem u_r, such that $u=(u_d,u_r)$ and $u_d=(u_1,\ldots,u_m)$, $u_r=(u_{m+1},\ldots,u_n)$. The dynamics of each subsystem is governed by

$$\dot{u}_d=f_d(u_d,u_r,\mu) , \tag{2}$$

$$\dot{u}_r=f_r(u_d,u_r,\mu) . \tag{3}$$

The desired chaotic orbit $\{u(0),u(1),\ldots\}$ may be obtained via a coevolving system sampled stroboscopically at equally spaced time intervals τ. In terms of the two subsystems the desired orbit is represented by $\{u_d(0),u_d(1),\ldots\}$ and $\{u_r(0),u_r(1),\ldots\}$. In order to lock the system onto the desired orbit, start the evolution of the system at $t=0$ with an initial condition $u'(0)=(u_d'(0),u_r'(0))$, which is slightly deviated from the desired orbit such that $u_d'(0)=u_d(0)$, but $u_r'(0)=u_r(0)+\delta u_r(0)$. The drive and the response variables now evolve according to the equations

$$\dot{u}_d'=f_d(u_d',u_r',\mu') , \tag{4}$$

$$\dot{u}_r'=f_r(u_d',u_r',\mu') . \tag{5}$$

At $t=\tau$, $u'(1)=(u_d'(1),u_r'(1))$. Set externally the drive part of the variable $u'(1)$ to the drive variable of the desired orbit so that $u_d'(1)=u_d(1)$ and u_r' is untouched. Further evolution takes place in a similar fashion with the drive variable being set to the drive variable of the desired orbit after each time step τ and the response variable is allowed to evolve freely.

In the limit of $\tau\rightarrow 0$ the above procedure reduces to the procedure for synchronization with the desired orbit proposed by Pecora and Carroll [1,4]. They have demonstrated that the system will settle down onto the desired orbit provided the subsystem Lyapunov exponents corresponding to the response variables are all negative. The SLE's of the response system are given by the eigenvalues (time averaged) of the $[(n-m)\times(n-m)]$-dimensional response subsystem Jacobian matrix J_r whose elements are given by

$$(J_r)_{ij}=\frac{\partial f_i(u_d,u_r',\mu)}{\partial u_j'} , \quad i,j=m+1,\ldots,n \tag{6}$$

where u_d are the values of the drive variables of the desired trajectory. The length of the transient after which the system settles down onto the desired orbit depends on the value of the largest SLE of the response system.

The Pecora-Carroll criterion for synchronization does not work for our finite-time-step procedure because during the time interval τ between two settings even the drive variables evolve freely and tend to drift away from the desired trajectory (see Fig. 1). Hence the SLE criterion for synchronization discussed above gets modified due to the finite size of the time step.

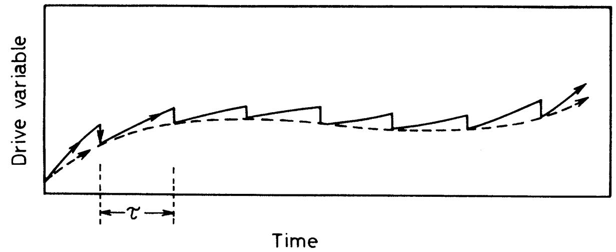

FIG. 1. A schematic diagram of the evolution of the drive variable as a function of time. The drive variable is set to the values of the desired trajectory after each time step τ. The evolution of the drive variable in the desired trajectory is shown by a dashed line.

III. SYNCHRONIZATION CRITERION FOR FINITE TIME STEP

In this section we obtain the criterion for synchronization for the finite-time-step method.

A. One-dimensional drive and one-dimensional response

Let us first consider the simple case of a one-dimensional drive and a one-dimensional response. Subtracting Eq. (2) from Eq. (4) and expanding to linear order we get

$$\Delta\dot{u}_d=f_d(u_d',u_r',\mu')-f_d(u_d,u_r,\mu) \simeq f_{dd}\Delta u_d+f_{dr}\Delta u_r+f_{d\mu}\Delta\mu , \tag{7}$$

where $f_{dd}=\partial f_d/\partial u_d$, $f_{dr}=\partial f_d/\partial u_r$, $f_{d\mu}=\partial f_d/\partial\mu$, $\Delta u_d=u_d'-u_d$, $\Delta u_r=u_r'-u_r$, and $\Delta\mu=\mu'-\mu$. Similarly, from Eqs. (3) and (5), we get

$$\Delta\dot{u}_r=f_r(u_d',u_r',\mu')-f_r(u_d,u_r,\mu) \simeq f_{rd}\Delta u_d+f_{rr}\Delta u_r+f_{r\mu}\Delta\mu , \tag{8}$$

where $f_{rd}=\partial f_r/\partial u_d$, $f_{rr}=\partial f_r/\partial u_r$, and $f_{r\mu}=\partial f_r/\partial\mu$. It is convenient to express the above equations in matrix form,

$$\begin{bmatrix}\Delta\dot{u}_d\\ \Delta\dot{u}_r\end{bmatrix}=\begin{bmatrix}f_{dd} & f_{dr}\\ f_{rd} & f_{rr}\end{bmatrix}\begin{bmatrix}\Delta u_d\\ \Delta u_r\end{bmatrix}+\begin{bmatrix}f_{d\mu}\\ f_{r\mu}\end{bmatrix}\Delta\mu . \tag{9}$$

When $\Delta\mu=0$ and assuming that the partial derivatives of f_d and f_r are time independent, Eq. (9) has the general solution

$$\begin{bmatrix}\Delta u_d\\ \Delta u_r\end{bmatrix}=X_1\begin{bmatrix}a\\ b\end{bmatrix}e^{\lambda_1 t}+X_2\begin{bmatrix}c\\ d\end{bmatrix}e^{\lambda_2 t} , \tag{10}$$

where

$$\lambda_1=\frac{f_{dd}+f_{rr}+D}{2} ,$$

$$\lambda_2=\frac{f_{dd}+f_{rr}-D}{2} ,$$

and $D=\sqrt{(f_{dd}-f_{rr})^2+4f_{dr}f_{rd}}$, X_1 and X_2 are constants, and a,b,c,d satisfy the equations

$$\frac{a}{b}=\frac{f_{dr}}{\lambda_1-f_{dd}}\ ,\quad \frac{c}{d}=\frac{f_{dr}}{\lambda_2+f_{dd}}\ . \tag{11}$$

When $\Delta\mu\neq 0$ the general solution is given by

$$\begin{bmatrix}\Delta u_d\\ \Delta u_r\end{bmatrix}=X_1\begin{bmatrix}a\\ b\end{bmatrix}e^{\lambda_1 t}+X_2\begin{bmatrix}c\\ d\end{bmatrix}e^{\lambda_2 t}+\begin{bmatrix}f_{d\mu}\\ f_{r\mu}\end{bmatrix}\left[\frac{e^{\lambda_1 t}-1}{\lambda_1}+\frac{e^{\lambda_2 t}-1}{\lambda_2}\right]\Delta\mu\ . \tag{12}$$

The constants X_1 and X_2 are determined by the initial condition $\Delta u_d(0)=0$ at $t=0$ and are given by

$$X_1=-\frac{c}{a}X_2\ ,\quad X_2=\frac{a\Delta u_r(0)}{ad-bc}\ .$$

After the first time step, $\Delta u_d(\tau)$ and $\Delta u_r(\tau)$ are given by

$$\Delta u_d(\tau)=\frac{ac\Delta u_r(0)}{ad-bc}[e^{\lambda_2\tau}-e^{\lambda_1\tau}]+\frac{f_{d\mu}}{f_{r\mu}}B\ , \tag{13}$$

$$\Delta u_r(\tau)=\Delta u_r(0)A+B\ , \tag{14}$$

where

$$A=\frac{ade^{\lambda_2\tau}-bce^{\lambda_1\tau}}{ad-bc}\ ,$$

$$B=f_{r\mu}\left[\frac{e^{\lambda_1\tau}-1}{\lambda_1}+\frac{e^{\lambda_2\tau}-1}{\lambda_2}\right]\Delta\mu\ .$$

At time $t=\tau$ we set $\Delta u_d(\tau)=0$. With the initial condition $u=[0,\Delta u_r(\tau)]$ and the evolution equation (12) we get the solution at time $t=2\tau$ to be

$$\Delta u_d(2\tau)=\frac{ac\Delta u_r(\tau)}{ad-bc}[e^{\lambda_2\tau}-e^{\lambda_1\tau}]+\frac{f_{d\mu}}{f_{r\mu}}B\ , \tag{15}$$

$$\begin{aligned}\Delta u_r(2\tau)&=\Delta u_r(\tau)A+B\\ &=\Delta u_r(0)A^2+B(A+1)\ .\end{aligned} \tag{16}$$

It is easy to see that at $t=n\tau$, the solution is given by

$$\Delta u_d(n\tau)=\frac{ac\Delta u_r[(n-1)\tau]}{ad-bc}[e^{\lambda_2\tau}-e^{\lambda_1\tau}]+\frac{f_{d\mu}}{f_{r\mu}}B\ , \tag{17}$$

$$\begin{aligned}\Delta u_r(n\tau)&=\Delta u_r[(n-1)\tau]A+B\\ &=\Delta u_r(0)A^n+B(A^{n-1}+\cdots+A+1)\\ &=\Delta u_r(0)A^n+B\frac{1-A^n}{1-A}\ .\end{aligned} \tag{18}$$

Let us first consider the case $\Delta\mu=0$. In this case $\Delta u_d(n\tau)$ and $\Delta u_r(n\tau)$ [Eqs. (17) and (18)] tend to zero provided $|A|<1$, i.e.,

$$|ade^{\lambda_2\tau}-bce^{\lambda_1\tau}|<1\ . \tag{19}$$

Hence asymptotically the driven variables u' perfectly synchronize with the desired orbit. The minimum value of A is obtained by $\partial A/\partial\tau=0$. This gives the condition for fastest convergence or the optimum value of τ. The condition simplifies to

$$e^{(\lambda_1-\lambda_2)\tau}=\frac{\lambda_1\lambda_2-\lambda_2 f_{rr}}{\lambda_1\lambda_2-\lambda_1 f_{rr}}\ . \tag{20}$$

In the case when $\Delta\mu\neq 0$, perfect synchronization is not possible. However, if $|A|<1$ [Eq. (19)], the variables u' will settle onto an orbit which has some correlation with the desired orbit. The degree of correlation depends on the magnitude of $\Delta\mu$ [1,4,7].

Consider the limit $\tau\to 0$. Using the relations (11) we get

$$\begin{aligned}A&\xrightarrow[\tau\to 0]{}1+\frac{ad\lambda_2-bc\lambda_1}{ad-bc}\tau+\cdots\\ &=1+f_{rr}\tau+\cdots\\ &\sim e^{f_{rr}\tau}\ .\end{aligned} \tag{21}$$

The finite-time-step criterion [Eq. (19)] implies that f_{rr}, which is the subsystem Lyapunov exponent, must be negative for observing synchronization. Thus the criterion for synchronization [Eq. (19)] reduces to the criterion proposed by Pecora and Carroll (see Sec. II) in the limit $\tau\to 0$.

B. Higher-dimensional systems

We now extend our analysis to an n-dimensional system subdivided into m drive variables and $n-m$ response variables. The linearized evolution equation (9) can now be written as

$$\Delta\dot{u}=J\Delta u+f_\mu\Delta\mu\ . \tag{22}$$

Here Δu is an n-dimensional column vector given by

$$\Delta u=\begin{bmatrix}\Delta u_d\\ \Delta u_r\end{bmatrix}, \tag{23}$$

where Δu_d and Δu_r are m- and $(n-m)$-dimensional column vectors, respectively. Similarly, f_μ is an n-dimensional column vector consisting of m- and $(n-m)$-dimensional column vectors $f_{d\mu}$ and $f_{r\mu}$. The matrix J is an $n\times n$ matrix given by

$$J=\begin{bmatrix}f_{dd} & f_{dr}\\ f_{rd} & f_{rr}\end{bmatrix}, \tag{24}$$

where J_{dd}, J_{dr}, J_{rd}, and J_{rr} are $m\times m$, $m\times(n-m)$, $(n-m)\times m$, and $(n-m)\times(n-m)$ matrices, respectively.

For the sake of simplicity we specialize to the case $\Delta\mu=0$. Equation (22) has the general solution

$$\Delta u=\sum_i^n X_i e^{\lambda_i t}v^{(i)}\ , \tag{25}$$

where λ_i and $v^{(i)}$ are the eigenvalues and the eigenvectors of the matrix J and the X_i's are constants to be evaluated using the initial condition at time $t=0$

$$U(0)=\sum_i^n X_i v^{(i)}\ , \tag{26}$$

where

$$U(t)=\begin{bmatrix}0\\ \Delta u_r(t)\end{bmatrix}. \tag{27}$$

Let V be an $n\times n$ matrix whose columns are the eigenvectors $v^{(i)}$,

$$V=\begin{bmatrix}v_1^{(1)} & v_1^{(2)} & \cdots & v_1^{(n)}\\ v_2^{(1)} & v_2^{(2)} & \cdots & v_2^{(n)}\\ \vdots & \vdots & \ddots & \vdots\\ v_n^{(1)} & v_n^{(2)} & \cdots & v_n^{(n)}\end{bmatrix}. \tag{28}$$

Let V^{-1} be the inverse of V. We define a column matrix $X(t)$ by the relation

$$X(t)=\begin{bmatrix}X_1e^{\lambda_1 t}\\ X_2e^{\lambda_2 t}\\ \vdots\\ X_ne^{\lambda_n t}\end{bmatrix}. \tag{29}$$

It is easy to see that Eq. (25) can be expressed in the form

$$\Delta u(t)=VX(t). \tag{30}$$

Thus the initial condition [Eq. (26)] at $t=0$ becomes

$$U(0)=VX(0). \tag{31}$$

Hence

$$X(0)=V^{-1}U(0). \tag{32}$$

It is useful to define a matrix W given by

$$W=\begin{bmatrix}e^{\lambda_1\tau}v_1^{(1)} & e^{\lambda_2\tau}v_1^{(2)} & \cdots & e^{\lambda_n\tau}v_1^{(n)}\\ e^{\lambda_1\tau}v_2^{(1)} & e^{\lambda_2\tau}v_2^{(2)} & \cdots & e^{\lambda_n\tau}v_2^{(n)}\\ \vdots & \vdots & \ddots & \vdots\\ e^{\lambda_1\tau}v_n^{(1)} & e^{\lambda_2\tau}v_n^{(2)} & \cdots & e^{\lambda_n\tau}v_n^{(n)}\end{bmatrix}. \tag{33}$$

We now rewrite the matrices V^{-1} and W in the following block matrix form:

$$V^{-1}=\begin{bmatrix}\tilde{v}_{dd} & \tilde{v}_{dr}\\ \tilde{v}_{rd} & \tilde{v}_{rr}\end{bmatrix}, \tag{34}$$

$$W=\begin{bmatrix}w_{dd} & w_{dr}\\ w_{rd} & w_{rr}\end{bmatrix}, \tag{35}$$

where the dimensions of the blocks are the same as the dimensions of the corresponding blocks of the matrix J [Eq. (24)]. We also define a projection matrix W_p which is obtained from W by setting the blocks w_{dd} and w_{dr} to zero:

$$W_p=\begin{bmatrix}0 & 0\\ w_{rd} & w_{rr}\end{bmatrix}. \tag{36}$$

After the first time step the solution is given by

$$\Delta u(\tau)=VX(\tau)=WX(0)=WV^{-1}U(0), \tag{37}$$

where we have used Eqs. (30), (28), and (33). We now set $\Delta u_d=0$. Thus the new initial condition is $U(\tau)$. Using the evolution equation it is easy to see that at the next time step $t=2\tau$ we have

$$\Delta u(2\tau)=WV^{-1}U(\tau)=WV^{-1}W_pV^{-1}U(0). \tag{38}$$

After n time steps we have

$$\Delta u(n\tau)=WV^{-1}(W_pV^{-1})^{n-1}U(0). \tag{39}$$

From Eq. (39) it is clear that the criterion for convergence of Δu, i.e., synchronization of u and u', is that the modulus of the eigenvalues of W_pV^{-1} should be less than one. The matrix W_pV^{-1} has the form

$$W_pV^{-1}=\begin{bmatrix}0 & 0\\ w_{rd}\tilde{v}_{dd}+w_{rr}\tilde{v}_{rd} & w_{rd}\tilde{v}_{dr}+w_{rr}\tilde{v}_{rr}\end{bmatrix}. \tag{40}$$

We note that m eigenvalues of W_pV^{-1} are zero. The remaining $(n-m)$ eigenvalues are determined by the solutions of the equation

$$|w_{rd}\tilde{v}_{dr}+w_{rr}\tilde{v}_{rr}-\lambda I|=0. \tag{41}$$

In the small-τ limit the matrices W and W_p can be expanded in the forms

$$W\xrightarrow[\tau\to 0]{}V+\tilde{\Lambda}\tau+\cdots, \tag{42}$$

where

$$\tilde{\Lambda}=\begin{bmatrix}\lambda_1v_1^{(1)} & \lambda_2v_1^{(2)} & \cdots & \lambda_nv_1^{(n)}\\ \lambda_1v_2^{(1)} & \lambda_2v_2^{(2)} & \cdots & \lambda_nv_2^{(n)}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_1v_n^{(1)} & \lambda_2v_n^{(2)} & \cdots & \lambda_nv_n^{(n)}\end{bmatrix} =\begin{bmatrix}\tilde{\Lambda}_{dd} & \tilde{\Lambda}_{dr}\\ \tilde{\Lambda}_{rd} & \tilde{\Lambda}_{rr}\end{bmatrix} \tag{43}$$

with the block partitioning being the same as for the matrix J [Eq. (24)] and

$$W_p\xrightarrow{\tau\to 0}V+\begin{bmatrix}0 & 0\\ \tilde{\Lambda}_{rd} & \tilde{\Lambda}_{rr}\end{bmatrix}\tau+\cdots. \tag{44}$$

Thus

$$\begin{aligned}&w_{rd}\tilde{v}_{dr}+w_{rr}\tilde{v}_{rr}\\ &\xrightarrow{\tau\to 0}v_{rd}\tilde{v}_{dr}+v_{rr}\tilde{v}_{rr}+(\tilde{\Lambda}_{rd}\tilde{v}_{dr}+\tilde{\Lambda}_{rr}\tilde{v}_{rr})\tau+\cdots\\ &=I+J_{rr}\tau+\tfrac{1}{2}(J_{rr}^2+J_{rd}J_{dr})\tau^2+\cdots,\end{aligned} \tag{45}$$

where we have used the relations $VV^{-1}=I$ and $\tilde{\Lambda}V^{-1}=J$. The term proportional to τ^2 is obtained in a similar fashion. Thus, in the small-τ limit the eigenvalues of J_{rr}, i.e., the subsystem Lyapunov exponents, decide the synchronization criterion. As the time step τ increases we get corrections due to the finite value of τ.

We have thus derived a criterion for synchronization which takes into account the fact that the system is set to the drive at finite intervals. This criterion reduces correctly to the usual criterion of negativity of the subsystem Lyapunov exponents in the τ tending to zero limit. The fact that the criterion is modified due to setting at finite intervals has interesting consequences which we will explore in the next section.

IV. EXAMPLES

We now illustrate the above analysis for the Lorentz equations [8]:

$$\dot{x}=\sigma(y-x)\ ,\quad \dot{y}=rx-xz-y\ ,\quad \dot{z}=xy-bz\ . \tag{46}$$

These equations show chaotic behavior for $r>\sigma(\sigma+b+3)/(\sigma-b-1)$. We have studied the synchronization with both the fixed point and the chaotic orbit. For a given value of τ and a given drive variable we obtain the largest eigenvalue Λ, of the matrix $W_p V^{-1}$ [Eq. (41)]. These eigenvalues are listed in Tables I, II, and III for the drive variables x, y, and z, respectively. They give us a measure of the rate at which the response trajectory approaches (or recedes from) the desired trajectory. If $\epsilon(0)$ is the distance between the two trajectories at the beginning of the iterations, then the separation between the two trajectories after n iterates is given by

$$\epsilon(T)=\Lambda^n\epsilon(0)\ , \tag{47}$$

where $T=n\tau$ is the transient time. We now fix the ratio $\epsilon(T)/\epsilon(0)=R$ and obtain the number of iterates and hence the total time T required for synchronization by a factor of R. These values of T are listed in the Tables I–III for different drive variables and $R=10^{-4}$. We also obtain the observed transient time $\bar{T}$ required for the separation between the trajectories to go down by the same factor R from actual numerical simulation of the procedure described in Sec. I and these transient times are again listed in the tables. It can be clearly seen that the set of values T and $\bar{T}$ agree very well.

We plot the transient time as a function of τ in Fig. 2 for x and y as the drive variables for synchronization with the fixed point. For small-τ values the transient time is almost a constant showing that the linear approximation in Eq. (45) is adequate and the synchronization criterion can be determined by the eigenvalues of J_{rr} or the subsystem Lyapunov exponents. As τ increases the effect of higher-order terms in Eq. (45) is felt and we start observing deviations from the linear behavior. The lowest-order departure from linear behavior is decided by the term $J_{rd}J_{dr}\tau^2/2$. We see that for x as the drive variable the transient time increases as τ increases, while for y as the drive variable it decreases as τ increases. From Tables I and II we see that similar behavior is observed for both the fixed point and the chaotic orbit.

We observe an interesting phenomena for z as the drive variable. For this case the largest SLE is marginal and hence synchronization is not expected according to the SLE criterion. However, we find that synchronization becomes possible due to the finite nature of the time step and the nonlinear correction discussed above. Table III gives the values of the transient times as a function of τ for synchronization with the fixed point and the same are plotted in Fig. 3. As τ increases, initially the transient time decreases almost exponentially, reaches a minimum, and then rises sharply. In no part of the graph is the behavior linear, as in Fig. 2, since the contribution of the linear term in Eq. (45) is zero and only the higher-order corrections contribute. The minimum of the transient time corresponds to an optimum choice of τ. (We have analyzed this situation before the one-dimensional case

TABLE I. The τ variation of the larger eigenvalue Λ of the matrix $W_p V^{-1}$, the transient times T from the eigenvalues, and the observed transient times $\bar{T}$ are listed. The values listed are those relevant for synchronization with the fixed point $x^*=y^*=-\sqrt{b(r-1)}$, $z^*=(r-1)$, and with the chaotic orbits of the Lorenz attractor. The drive variable is x and the parameter values are $\sigma=10.0$, $b=8/3$, $r=60.0$.

	Fixed point			Chaotic orbit		
τ	Λ	T	$\bar{T}$	Λ	T	$\bar{T}$
0.05	0.929 13	6.25	6.20	0.933 70	6.70	10.75
0.02	0.965 83	5.28	5.32	0.968 85	5.82	5.86
0.01	0.982 19	5.12	5.14	0.983 46	5.22	5.35
0.005	0.990 95	5.07	5.07	0.991 04	5.11	5.62
0.002	0.996 35	5.04	5.03	0.996 23	4.87	5.12
0.001	0.998 17	5.03	5.03	0.998 13	4.93	5.03
0.0005	0.999 08	5.03	5.03	0.999 07	4.96	5.12
0.0002	0.999 63	5.03	5.03	0.999 63	4.96	5.11
0.0001	0.999 82	5.03	5.03	0.999 81	4.96	5.02

TABLE II. Eigenvalues Λ and transient times T and $\bar{T}$ for the same case as Table I, but with a y drive.

	Fixed point			Chaotic orbit		
τ	Λ	T	$\bar{T}$	Λ	T	$\bar{T}$
0.05	0.673 45	1.15	1.15	0.718 17	1.40	1.45
0.02	0.868 89	1.30	1.30	0.918 96	2.18	1.90
0.01	0.950 03	1.79	1.81	0.966 23	2.68	2.340
0.005	0.981 71	2.49	2.50	0.984 82	3.01	2.90
0.002	0.993 92	3.02	3.02	0.994 38	3.27	3.19
0.001	0.997 15	3.23	3.23	0.997 28	3.39	3.31
0.0005	0.998 62	3.34	3.34	0.998 66	3.46	3.38
0.0002	0.999 46	3.41	3.41	0.999 47	3.48	3.42
0.0001	0.999 73	3.43	3.43	0.999 73	3.49	3.43

TABLE III. Eigenvalues Λ and transient times T and $\bar{T}$ for the same case as Table I, but with a z drive and for the fixed point case alone.

	Fixed point		
τ	Λ	T	$\bar{T}$
0.05	0.699 97	1.25	1.25
0.02	0.932 89	2.64	2.65
0.01	0.984 83	6.02	6.03
0.005	0.996 33	12.52	12.52
0.002	0.999 42	31.86	31.86
0.001	0.999 86	64.07	64.06
0.0005	0.999 96	128.47	128.46
0.0002	0.999 994	321.65	
0.0001	0.999 998	643.62	

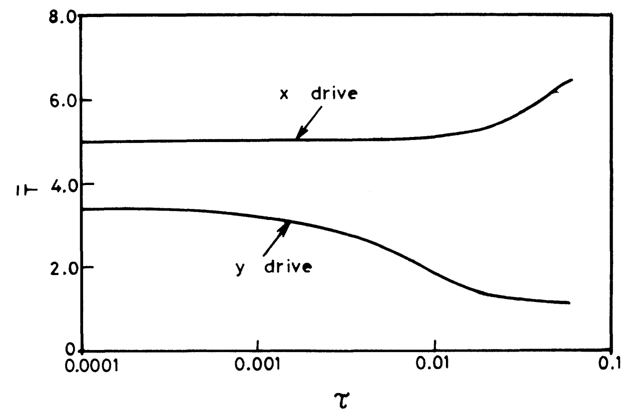

FIG. 2. The observed transient time $\bar{T}$ as a function of τ for synchronization with the fixed point $x^*=y^*=-\sqrt{b(r-1)}$, $z^*=(r-1)$ for the Lorenz system at the parameter values $\sigma=10.0$, $b=8/3$, and $r=60.0$, with x and y as the drive variables.

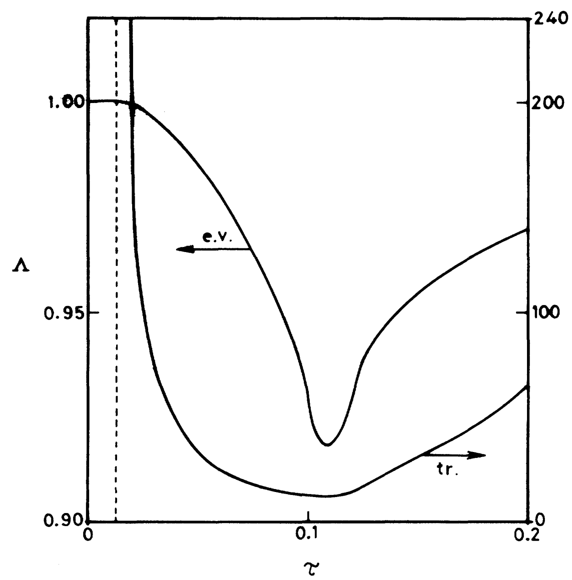

FIG. 4. The larger eigenvalue Λ of the matrix $W_p V^{-1}$ as a function of τ for synchronization with the fixed point $z^*=-y^*$, $x^*=-ay^*$, $y^*=(-c+\sqrt{c^2-4ab})/2a$ of the Rössler system at the parameter values $a=0.2$, $b=0.2$, and $c=6.0$ with z as the drive variable. We also plot the observed transient time $\bar{T}$ as a function of τ on the same graph. The vertical dashed line represents the asymptote of the transient time where the eigenvalue $\Lambda=1.0$. The eigenvalues are plotted on the scale to the left while the transient times are plotted on the scale to the right.

[see Eq. (20)].) We have also observed that synchronization with chaotic orbits is possible for z as the drive variable and τ values around 0.01. However, we have not been able to compare the observed transient time with the transient time obtained from the eigenvalues since the eigenvalues could not be determined to a sufficient accuracy in this case.

The second system for which we study the effect of finite time step is the Rössler system [9] given by

$$\dot{x}=-y-z\ ,\quad \dot{y}=ay+x\ ,\quad \dot{z}=b+xz-cz\ . \tag{48}$$

Consider the case of synchronization with the fixed point $z^*=-y^*$, $x^*=-ay^*$, and $y^*=(-c+\sqrt{c^2-4ab})/2a$. According to the SLE criterion the only case for which synchronization is possible is for the case where the drive variable is y. However, for a finite value of τ, we find that the solution synchronizes for y and z as the drive variables. For y as the drive variable, the values of the transient time show a behavior similar to the Lorenz system with y drive, i.e., the transient time decreases as τ increases. There is good agreement between the observed transient times and the transient times obtained from the eigenvalues.

We see an interesting phenomenon for the case where the drive variable is z. We plot the behavior of the largest eigenvalue Λ of the matrix $W_p V^{-1}$ as a function of τ in Fig. 4. The behavior of the observed transient time $\bar{T}$ is a function of τ is plotted on the same graph. the transient times T estimated from the eigenvalues agree very well with the observed transient times $\bar{T}$. The largest eigenvalue Λ starts off with the value 1.0 at $\tau=0.0$, rises above 1.0 with increasing τ, then again decreases and crosses 1.0 at the value $\tau=0.0133\ldots$ to reach a minimum around $\tau=0.11$, and rises again. The transient time $\bar{T}$ appears to diverge in the neighborhood of $\tau=0.0133\ldots$, where the eigenvalue Λ crosses 1.0, decreases with increasing τ, reaches a minimum around $\tau=0.11$, and rises again. This minimum should correspond to the optimum choice of τ as in the Lorenz case. It is easy to see that although the minima of the eigenvalue and the transient time are not the same, they will be close to each other.

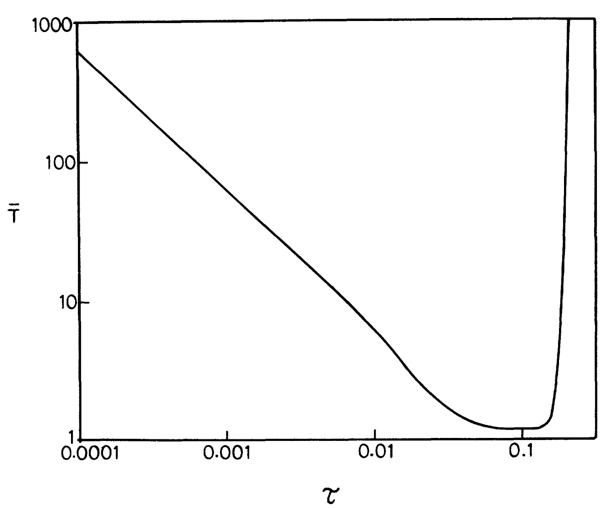

FIG. 3. The observed transient time $\bar{T}$ as a function of τ for synchronization with the fixed point $x^*=y^*=-\sqrt{b(r-1)}$, $z^*=(r-1)$ for the Lorenz system at the parameter values $\sigma=10.0$, $b=8/3$, and $r=60.0$, with z as the drive.

V. DISCUSSION AND CONCLUSION

We have shown that synchronization of chaotic orbits is possible using a finite-time-step method. We have obtained a criterion of synchronization for this method. This criterion reduces to the SLE criterion of Pecora and Carroll in the limit $\tau \to 0$. Using the finite-time-step procedure it is possible to observe synchronization even in cases where the possibility of synchronization is ruled out by the SLE criterion. We have demonstrated this by the examples of the Lorenz and Rössler systems where synchronization is observed with z as the drive variable. In the case of the Lorenz system we have a marginal SLE or the eigenvalue $\Lambda = 1.0$, which is pulled down below 1.0 because of the finite time step. In the case of the Rössler system the largest SLE is positive (i.e., $\Lambda > 1.0$) and the finite time step not only compensates for this positive SLE but leads to synchronization for large values of τ. We have also seen that it is possible to obtain an optimum choice of τ which gives minimum transient time and hence fastest convergence.

Thus the finite-time-step method has proved to be successful in achieving synchronization in at least two cases where the method of continuous setting fails. The reason for this success is apparent from Eq. (45). The lowest-order correction to the SLE criterion is given by the term $\frac{1}{2} J_{rd} J_{dr} \tau^2$. This term includes the effect of the drive variables as well as the response variables as the drive variables also evolve freely between two settings of the drive in this method. A rough rule of thumb for the rate of convergence can be obtained as follows. This rate depends on the angle, say θ, made by the drive direction with the direction along which the Lyapunov exponent is the largest (i.e., the direction corresponding to the maximum stretching). If this Lyapunov exponent has a value $\lambda_{\max}$, the length of the transient is controlled by the factor $\sin^2\theta \exp(\lambda_{\max}\tau)$. The length of the transient decreases with decrease in θ. Thus we expect that the finite-time-step method will give better convergence where the drive variable makes a small angle with the direction of maximum stretching on an average. Such a situation might occur in several systems.

We thus see that the finite-time-step method for synchronization can be advantageous for systems of the type described above. An optimum choice of τ may also be possible in such cases. Since experimental realizations of such systems should be possible, our analysis may prove to be useful in a variety of practical contexts.

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